Kinematics & Dynamics of Linkages

Lecture 18: Cam Design

Polynomial functions

General polynomial form:

$$s = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots + C_n x^n$$

s = displacement

x = independent variable replaced by ϑ/β or time t

C_n = unknown constants

n^{th} degree polynomial has n+1 terms

the number of boundary conditions determines the order

Polynomial functions

- Displacement, velocity, acceleration and jerk

$$s = C_0 + C_1\left(\frac{\theta}{\beta}\right) + C_2\left(\frac{\theta}{\beta}\right)^2 + \ldots\ldots + C_n\left(\frac{\theta}{\beta}\right)^n$$

$$v = \frac{ds}{d\theta} = \frac{1}{\beta}\left[C_1 + 2C_2\left(\frac{\theta}{\beta}\right) + 3C_3\left(\frac{\theta}{\beta}\right)^2 + \ldots\ldots + nC_n\left(\frac{\theta}{\beta}\right)^{n-1}\right]$$

$$a = \frac{d^2 s}{d\theta^2} = \frac{1}{\beta^2}\left[2C_2 + 6C_3\left(\frac{\theta}{\beta}\right) + \ldots\ldots + n(n-1)C_n\left(\frac{\theta}{\beta}\right)^{n-2}\right]$$

$$j = \frac{d^3 s}{d\theta^3} = \frac{1}{\beta^3}\left[6C_3 + n(n-1)(n-2)C_n\left(\frac{\theta}{\beta}\right)^{n-3}\right]$$

Double Dwell Case 1

Boundary conditions

Disp, velocity and acceleration

Rise: 6 boundary conditions

$$\theta = 0 \quad \Rightarrow \quad s = 0 \quad v = 0 \quad a = 0$$

$$\theta = \beta_1 \quad \Rightarrow \quad s = h \quad v = 0 \quad a = 0$$

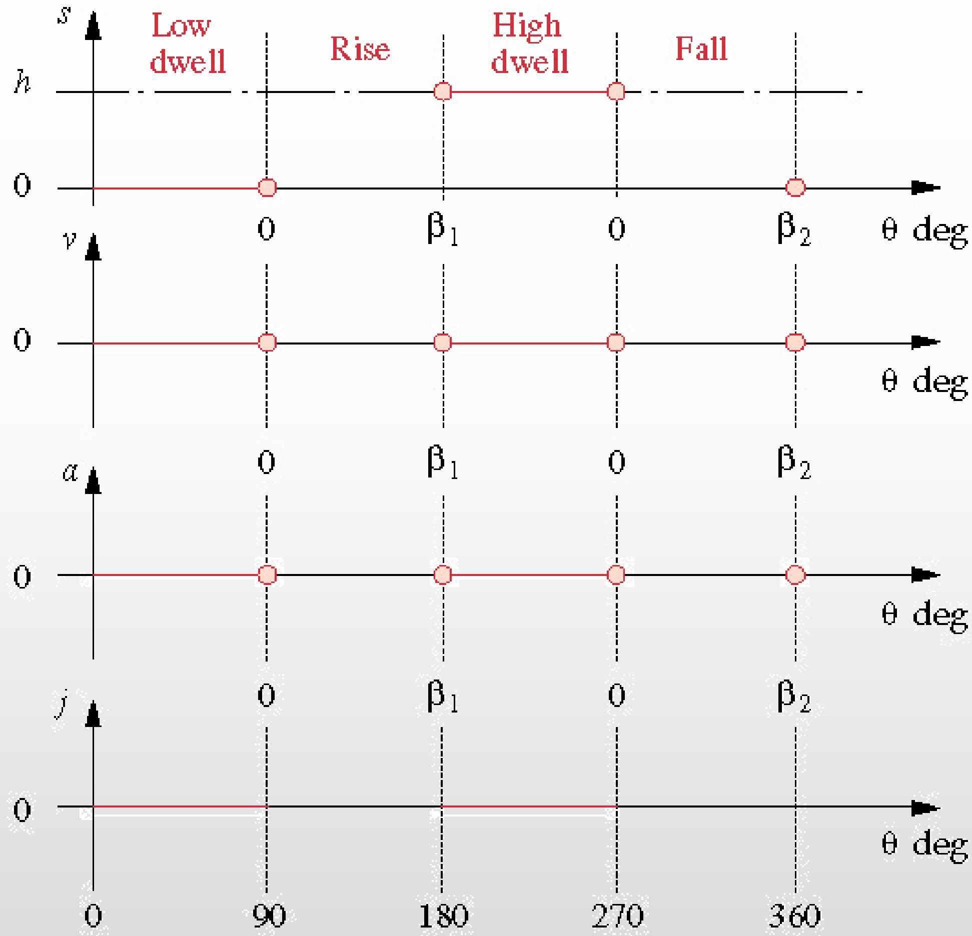
Fall: 6 boundary conditions

$$\theta = 0 \quad \Rightarrow \quad s = h \quad v = 0 \quad a = 0$$

$$\theta = \beta_2 \quad \Rightarrow \quad s = 0 \quad v = 0 \quad a = 0$$

We can solve for 6 constants only

$$C_0, C_1, C_2, C_3, C_4 \ \mathbf{and} \ C_5$$



Rise – Left side boundary conditions

- Solve for rise: $\theta = 0 \quad \rightarrow \quad s = 0 \quad v = 0 \quad a = 0$

$$s = C_0 + C_1\left(\frac{\theta}{\beta}\right) + C_2\left(\frac{\theta}{\beta}\right)^2 + \ldots\ldots + C_5\left(\frac{\theta}{\beta}\right)^5 \quad \rightarrow \quad C_0 = 0$$

$$v = \frac{ds}{d\theta} = \frac{1}{\beta}\left[C_1 + 2C_2\left(\frac{\theta}{\beta}\right) + 3C_3\left(\frac{\theta}{\beta}\right)^2 + \ldots\ldots + 5C_5\left(\frac{\theta}{\beta}\right)^4\right] \quad \rightarrow \quad C_1 = 0$$

$$a = \frac{d^2s}{d\theta^2} = \frac{1}{\beta^2}\left[2C_2 + 6C_3\left(\frac{\theta}{\beta}\right) + \ldots\ldots + 20C_5\left(\frac{\theta}{\beta}\right)^3\right] \quad \rightarrow \quad C_2 = 0$$

Rise – Right side boundary conditions

- Solve for rise: $\theta = \beta_1 \quad \Rightarrow \quad s = h \quad v = 0 \quad a = 0$

$$s = C_3\left(\frac{\theta}{\beta}\right)^3 + C_4\left(\frac{\theta}{\beta}\right)^4 + C_5\left(\frac{\theta}{\beta}\right)^5 \quad \rightarrow \quad C_3 + C_4 + C_5 = h$$

$$v = \frac{ds}{d\theta} = \frac{1}{\beta}\left[3C_3\left(\frac{\theta}{\beta}\right)^2 + 4C_4\left(\frac{\theta}{\beta}\right)^3 + 5C_5\left(\frac{\theta}{\beta}\right)^4\right] \quad \rightarrow \quad 3C_3 + 4C_4 + 5C_5 = 0$$

$$a = \frac{d^2s}{d\theta^2} = \frac{1}{\beta^2}\left[6C_3\left(\frac{\theta}{\beta}\right) + 12C_4\left(\frac{\theta}{\beta}\right)^2 + 20C_5\left(\frac{\theta}{\beta}\right)^3\right] \quad \rightarrow \quad 6C_3 + 12C_4 + 20C_5 = 0$$

Solving the Rise equations

- Equations

$$C_3 + C_4 + C_5 = h$$

$$3C_3 + 4C_4 + 5C_5 = 0$$

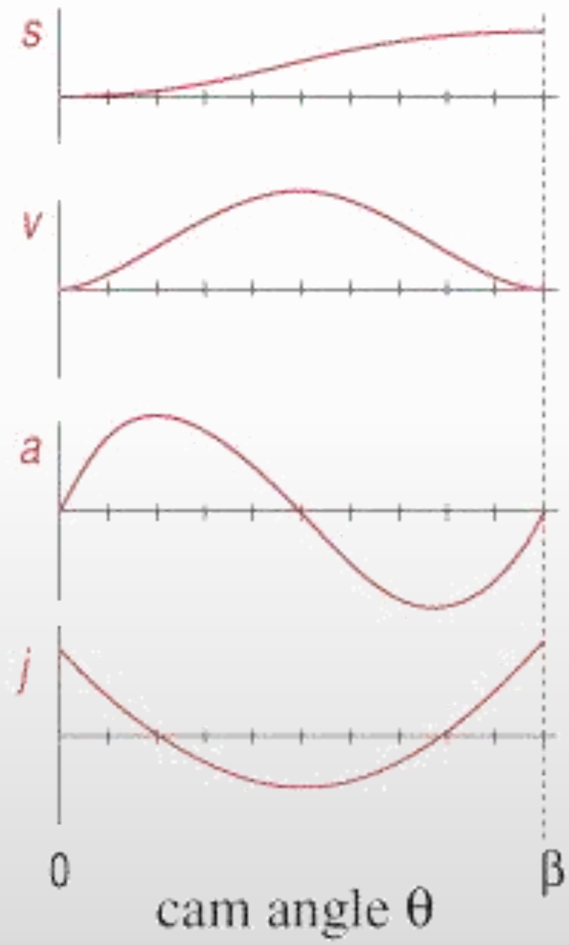
$$6C_3 + 12C_4 + 20C_5 = 0$$

- Solution

$$C_3 = 10h \qquad C_4 = -15h \qquad C_5 = 6h$$

- Called 3-4-5 polynomial rise

$$s = h\left[10\left(\frac{\theta}{\beta}\right)^3 - 15\left(\frac{\theta}{\beta}\right)^4 + 6\left(\frac{\theta}{\beta}\right)^5\right]$$



Fall – Left side boundary conditions

- Solve for the constants C_0, C_1, C_2, C_3, C_4 and C_5

$$\theta = 0 \quad \Rightarrow \quad s = h \quad v = 0 \quad a = 0$$

$$\theta = \beta_2 \quad \Rightarrow \quad s = 0 \quad v = 0 \quad a = 0$$

$$s = C_0 + C_1\left(\frac{\theta}{\beta}\right) + C_2\left(\frac{\theta}{\beta}\right)^2 + C_3\left(\frac{\theta}{\beta}\right)^3 + C_4\left(\frac{\theta}{\beta}\right)^4 + C_5\left(\frac{\theta}{\beta}\right)^5$$

- Called 3–4–5 polynomial fall

$$s = h - h\left[10\left(\frac{\theta}{\beta}\right)^3 - 15\left(\frac{\theta}{\beta}\right)^4 + 6\left(\frac{\theta}{\beta}\right)^5\right]$$

Double Dwell Case 2

Boundary conditions

Disp, velocity, acceleration, jerk

Rise: 8 boundary conditions

$$\theta = 0 \quad \Rightarrow \quad s = 0 \quad v = 0 \quad a = 0 \quad j = 0$$

$$\theta = \beta_1 \quad \Rightarrow \quad s = h \quad v = 0 \quad a = 0 \quad j = 0$$

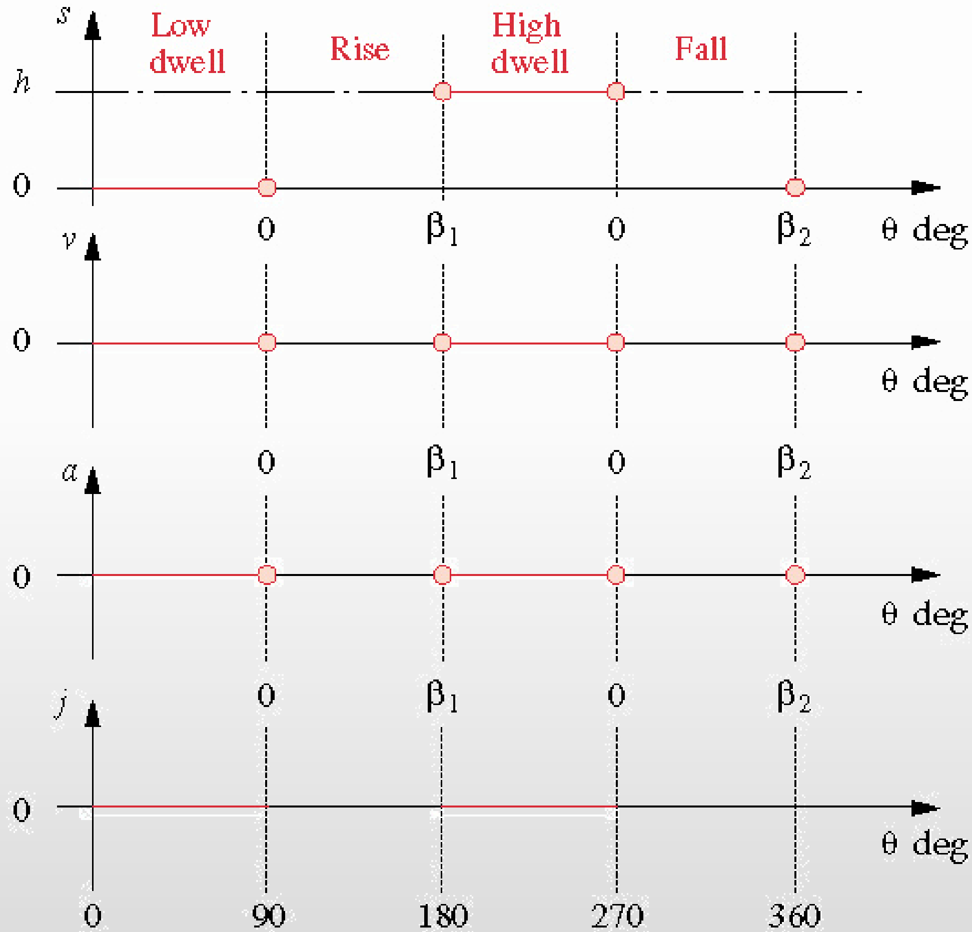
Fall: 8 boundary conditions

$$\theta = 0 \quad \Rightarrow \quad s = h \quad v = 0 \quad a = 0 \quad j = 0$$

$$\theta = \beta_2 \quad \Rightarrow \quad s = 0 \quad v = 0 \quad a = 0 \quad j = 0$$

We can solve for 8 constants

$C_0, C_1, C_2, C_3, C_4, C_5, C_6$ **and** C_7



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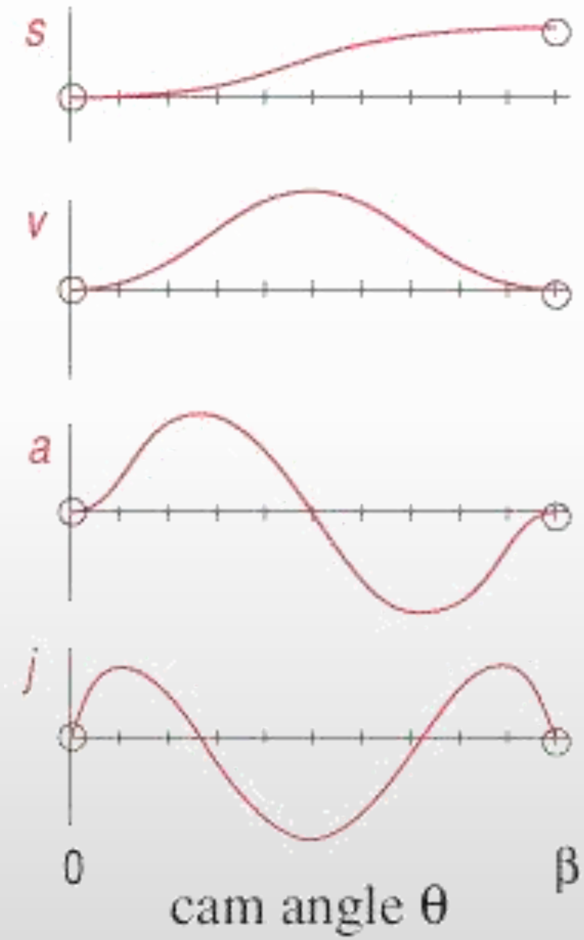
Solution

- Solve for rise : Follow steps for 3–4–5 polynomial and solve for

$$C_0, C_1, C_2, C_3, C_4, C_5, C_6 \ \mathbf{and} \ C_7$$

- Called 4–5–6–7 polynomial rise

$$s = h\left[35\left(\frac{\theta}{\beta}\right)^4 - 84\left(\frac{\theta}{\beta}\right)^5 + 70\left(\frac{\theta}{\beta}\right)^6 - 20\left(\frac{\theta}{\beta}\right)^7\right]$$



Example

Consider the cam design specifications below:

Dwell : at 0 displacement for 90°

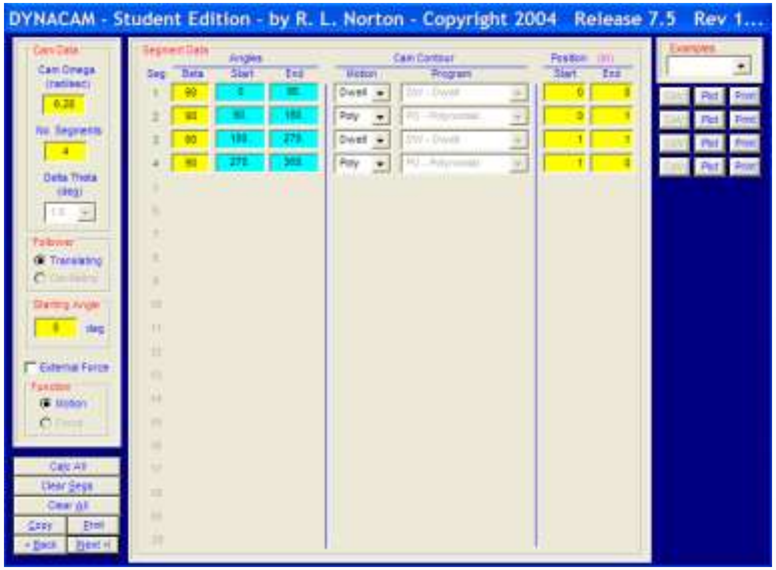
Rise : 1" in 90°

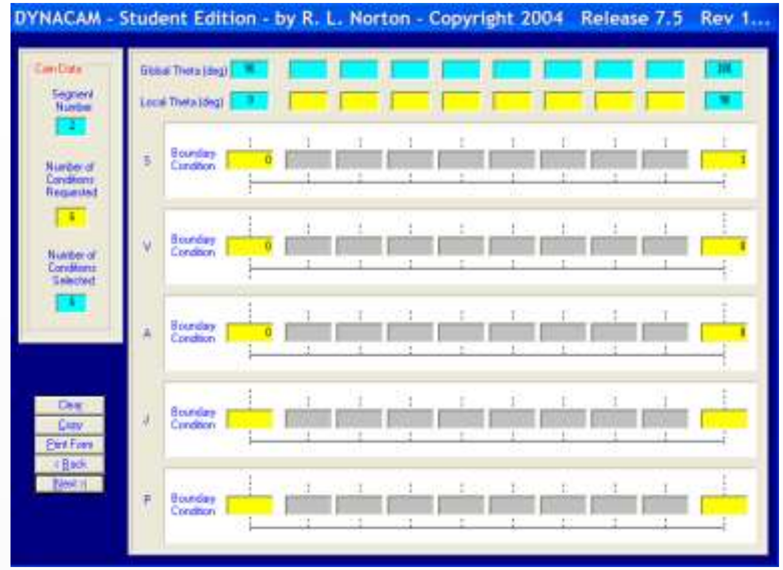
Dwell : at 1" displacement for 90°

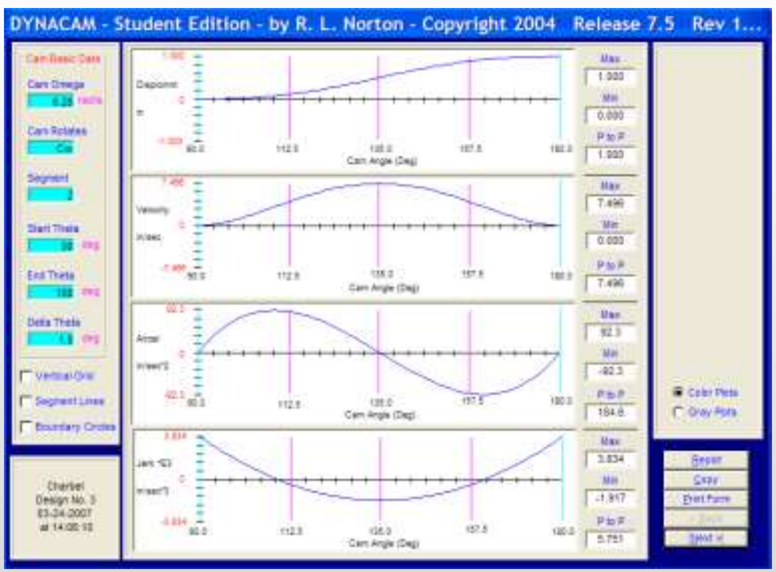
Fall : 1" in 90°

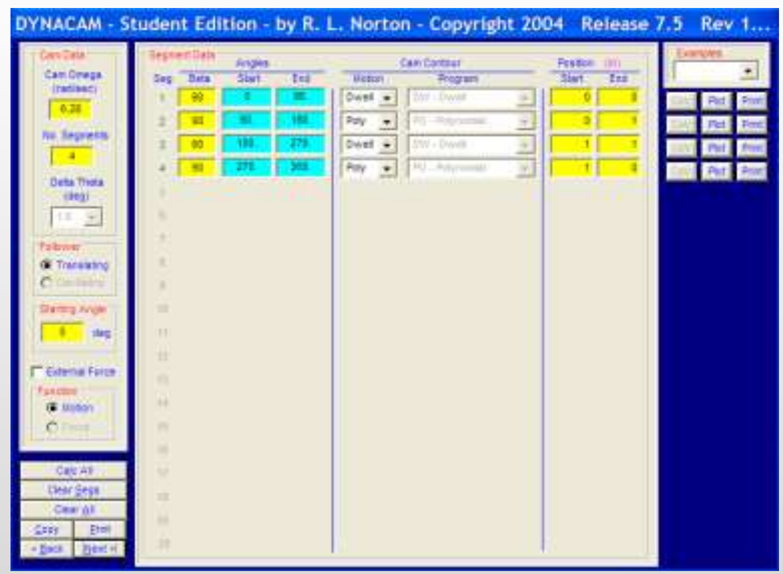
ω : 2π rad/s

Solution DYNACAM

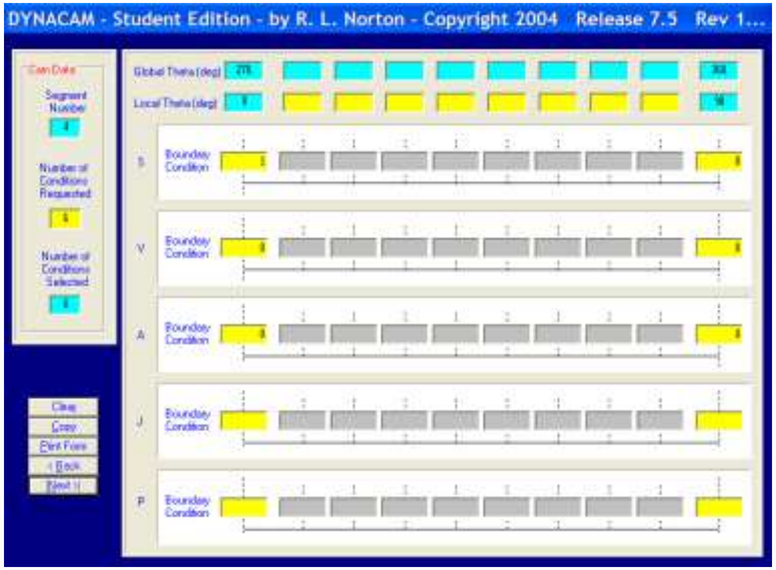


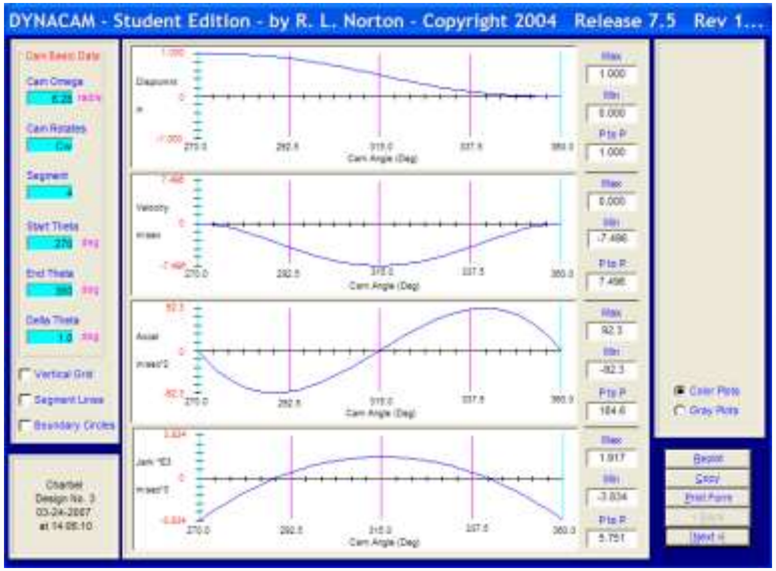


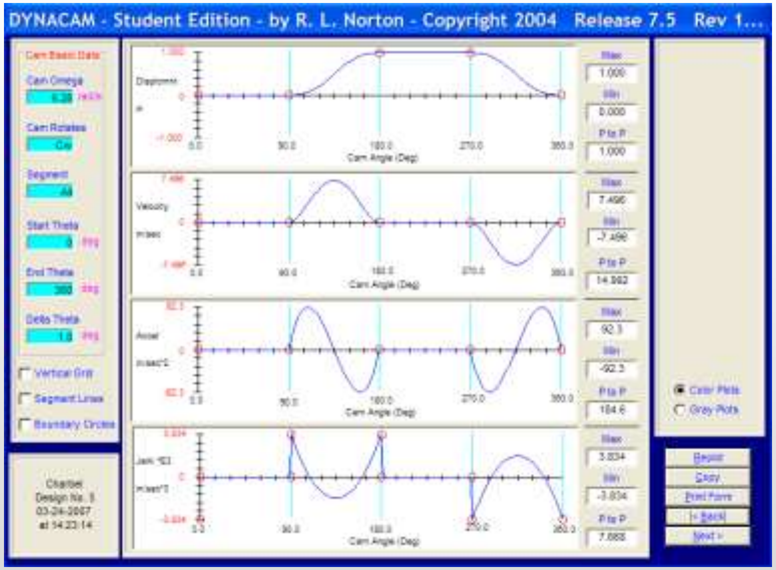


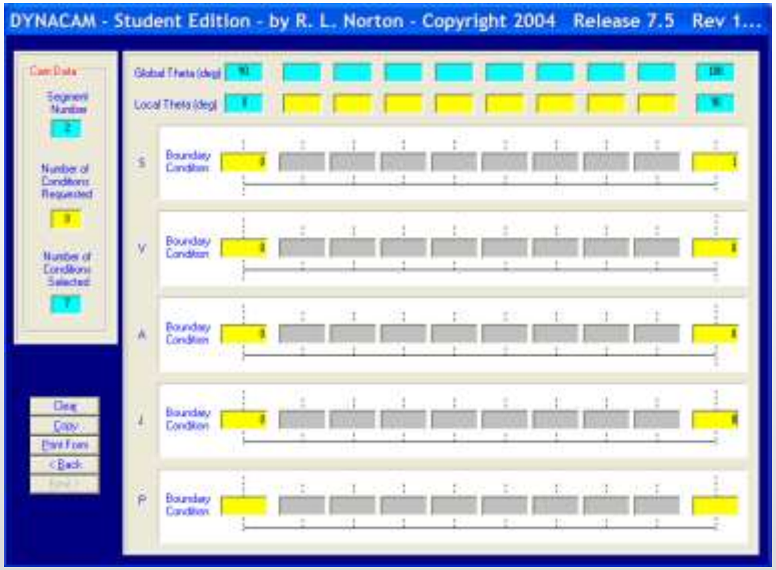


Solution DYNACAM

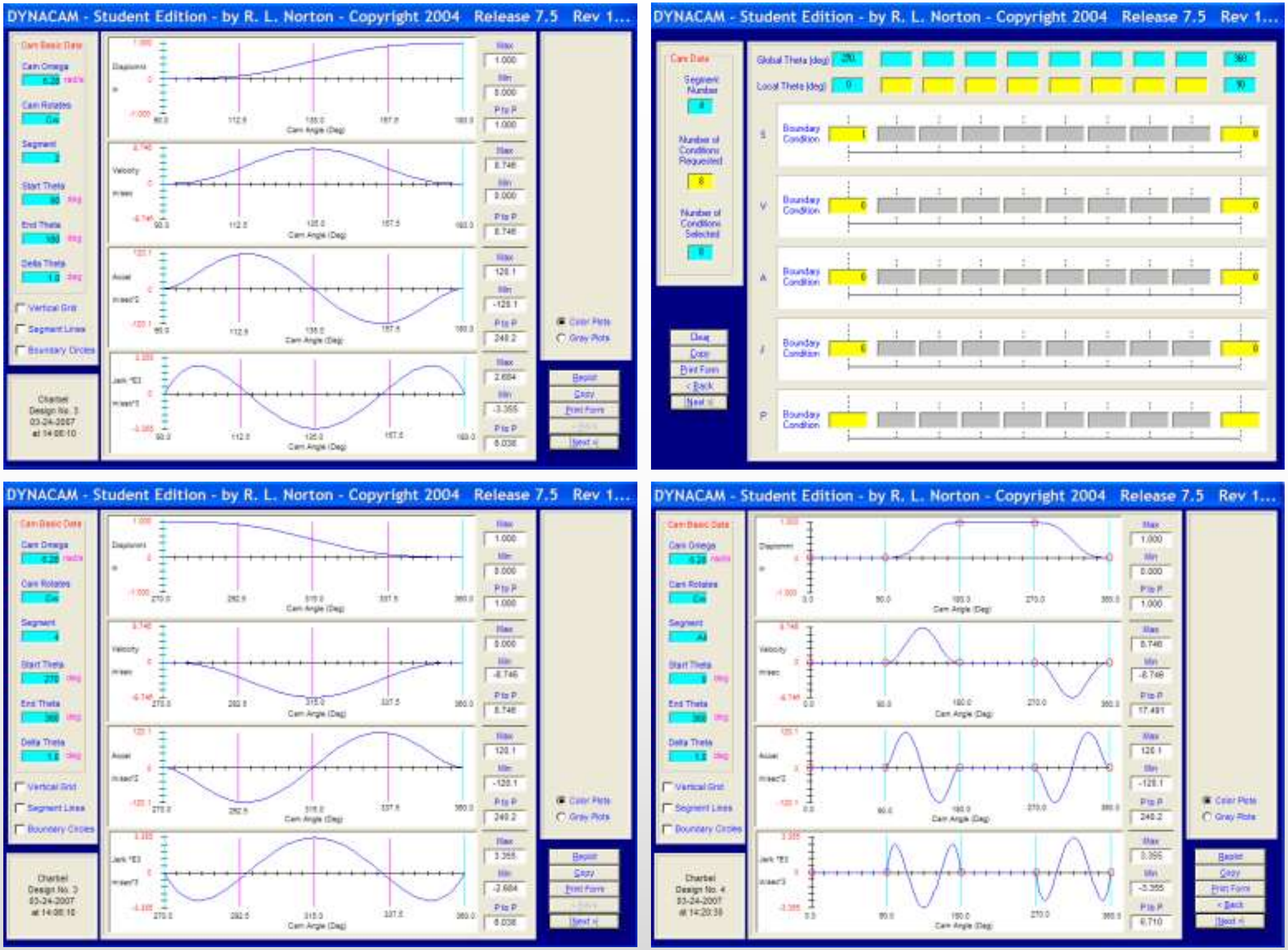






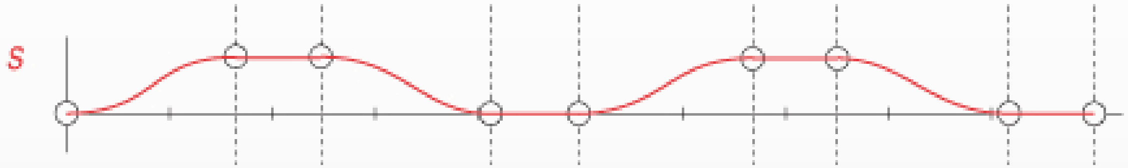


Solution DYNACAM



Cam Design

- Polynomials are good functions for displacements as long as they are of degree 5 and higher.



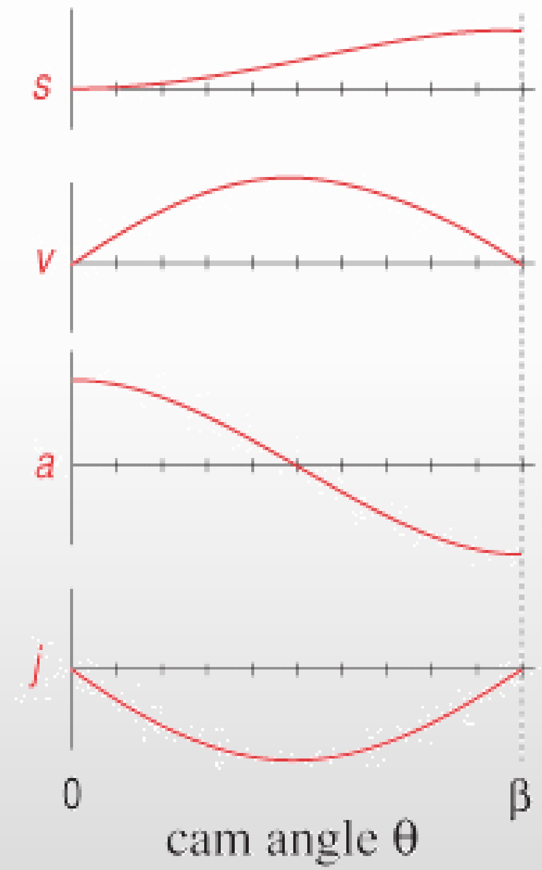
- What about the harmonic functions?
  - They remain continuous throughout any number of differentiations
  - Sine ⟶ cosine ⟶ −sine ⟶ −cosine ⟶ ……
  - 90º phase shifts of the functions

Harmonic functions

- Simple Harmonic Motion (SHM)
- Cycloidal Displacement
- Combined functions
  - Constant acceleration
  - Trapezoidal acceleration
  - Modified trapezoidal acceleration
    - Modified sinusoidal acceleration

Simple Harmonic Motion (SHM)

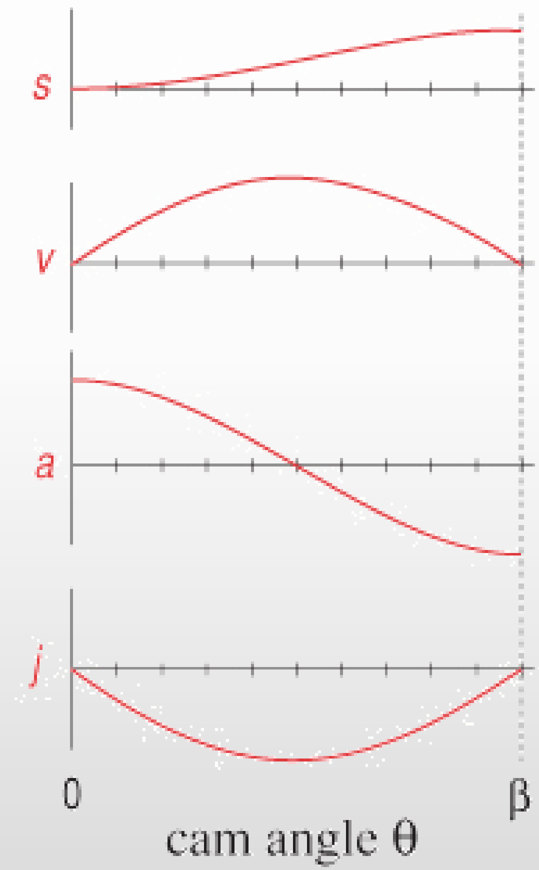
- Velocity is continuous – zero at ends
- Acceleration is not continuous
  - Non-zero start and finish values
- Dwells will result in zero acceleration at ends
  - Discontinuities exist which
  - This results in infinite spikes in the jerk function
- Sine functions (SHM) does not work
- It will work only for RF (180^0–180^0)



Simple Harmonic Motion (SHM)

$$s = \frac{h}{2}\left[1 - \cos\left(\pi \frac{\theta}{\beta}\right)\right] \qquad v = \frac{\pi}{\beta}\frac{h}{2}\sin\left(\pi \frac{\theta}{\beta}\right)$$

$$a = \frac{\pi^2}{\beta^2}\frac{h}{2}\cos\left(\pi \frac{\theta}{\beta}\right) \qquad j = -\frac{\pi^3}{\beta^3}\frac{h}{2}\sin\left(\pi \frac{\theta}{\beta}\right)$$



Where: h = total rise

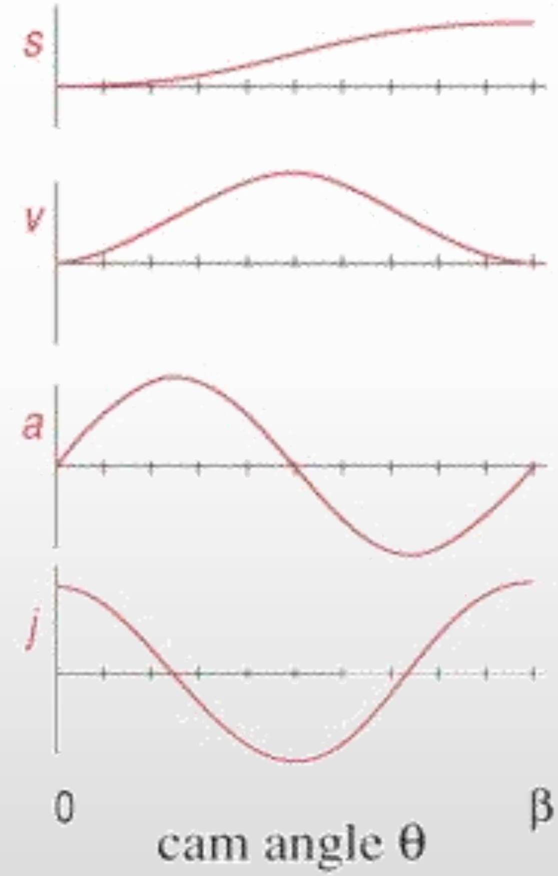
θ = camshaft angle

β = total angle of the rise interval

□ $\leq \theta \leq \beta$

Cycloidal displacement

Cycloidal functions solve the problem of discontinuity by starting with the acceleration and integrating to get S and V curves.



$$s = h\left[\frac{\theta}{\beta} - \frac{1}{2\pi}\sin\left(2\pi\frac{\theta}{\beta}\right)\right] \qquad v = \frac{h}{\beta}\left[1-\cos\left(2\pi\frac{\theta}{\beta}\right)\right]$$

$$a = 2\pi\frac{h}{\beta^2}\sin\left(2\pi\frac{\theta}{\beta}\right) \qquad j = 4\pi^2\frac{h}{\beta^3}\cos\left(2\pi\frac{\theta}{\beta}\right)$$

Combined Functions

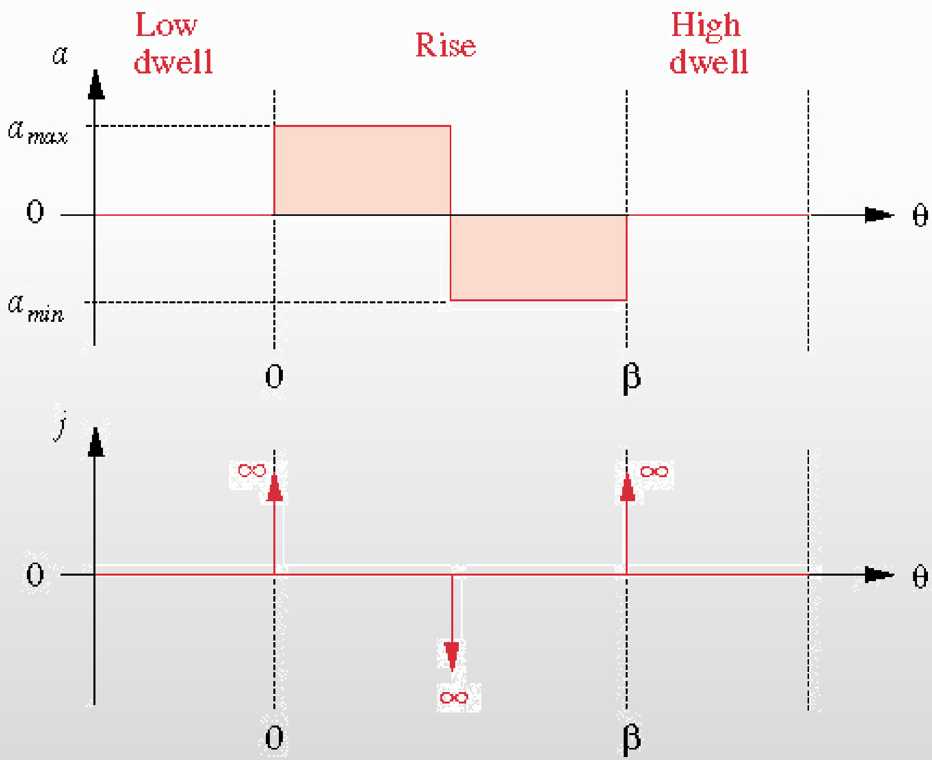
- Dynamic force is proportional to acceleration: $F = ma$
  - minimize acceleration → minimize dynamic force
- Kinetic energy is proportional to velocity
  - minimize velocity → minimize stored kinetic energy
- We want to choose a function that satisfies our constraints

Combined Functions Types

- Constant acceleration
- Trapezoidal acceleration
- Modified Trapezoidal acceleration
- Modified Sinusoidal acceleration

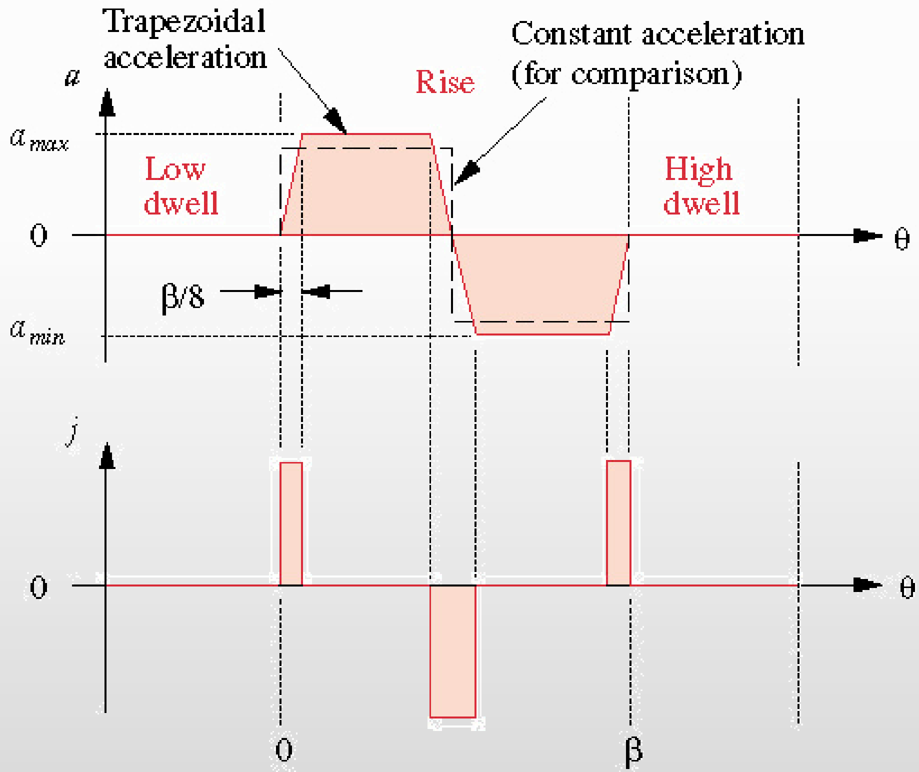
Combined Functions: Constant acceleration

minimize the peak value of acceleration > use square wave > discontinuities



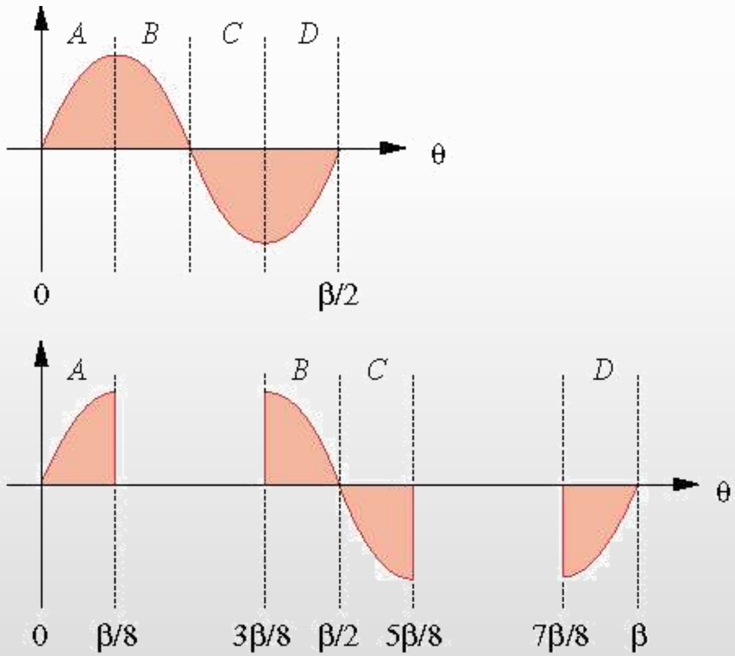
Combined Functions: Trapezoidal acceleration

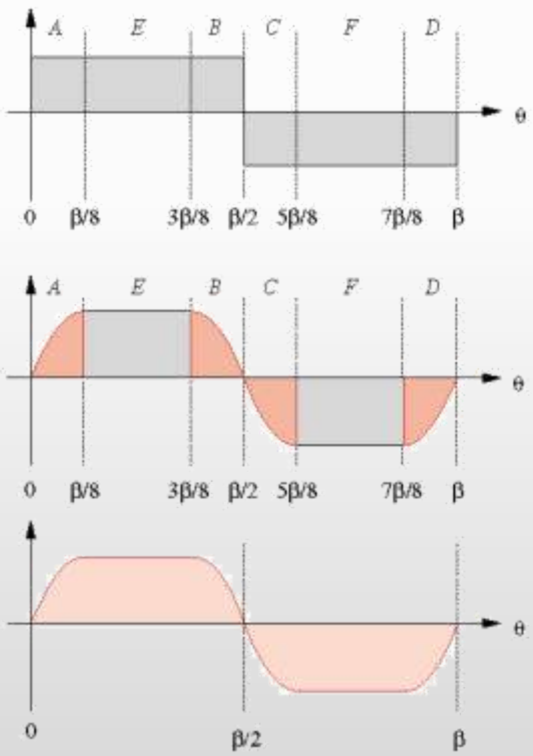
modify the corners of the square wave > discontinuous jerk > vibrations



Combined Functions: Modified Trapezoidal acc.

modify the corners of the square wave using sine > Low peak acceleration





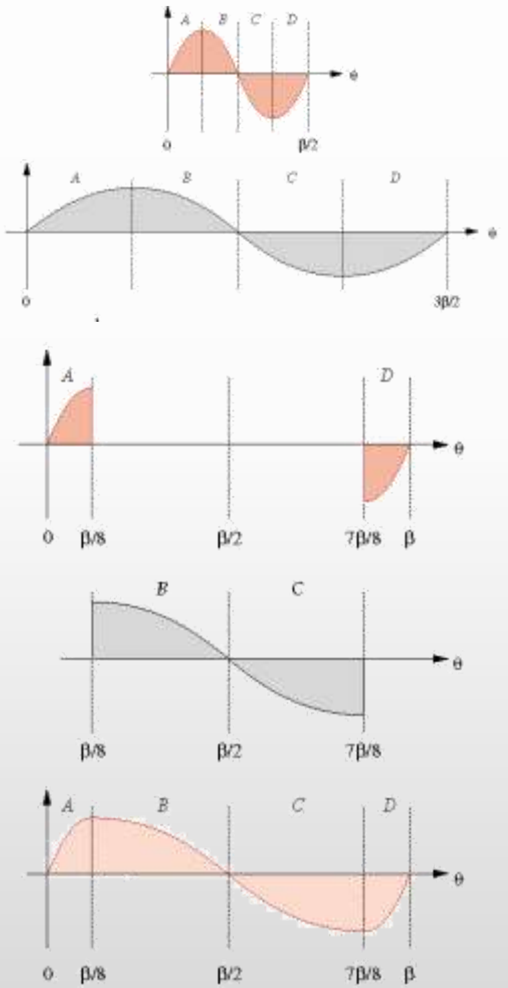
Combined Functions: Modified sinusoidal acc.

Sine acceleration curve – cycloidal displacement

- Has smooth jerk curve compared to modified trapezoidal
- Has higher peak acceleration

Combine two harmonic (sinusoid)

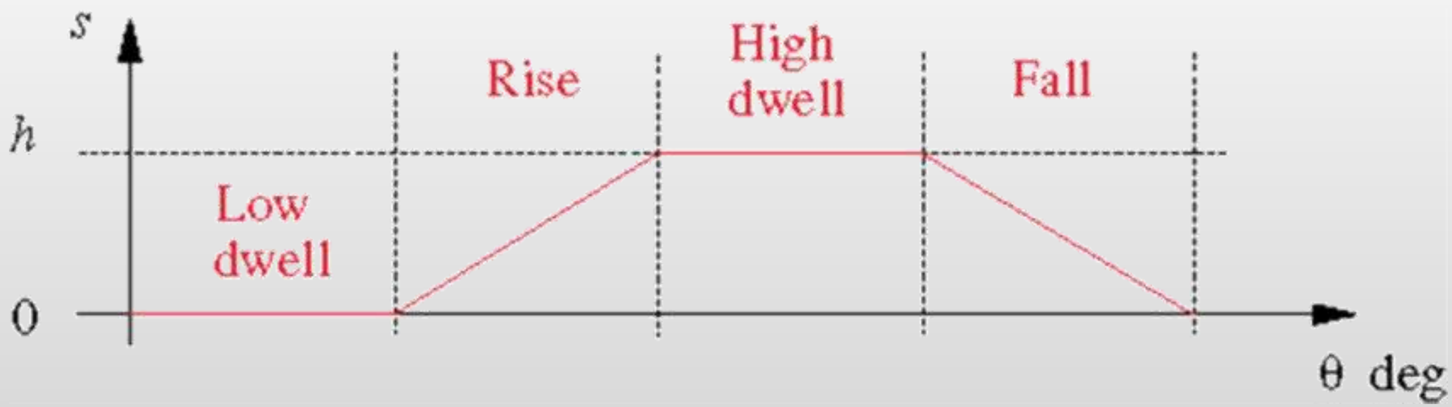
- Reduce the peak acceleration
- Peak velocity is lower
- Most commonly used for double dwell cams



SCCA double dwell

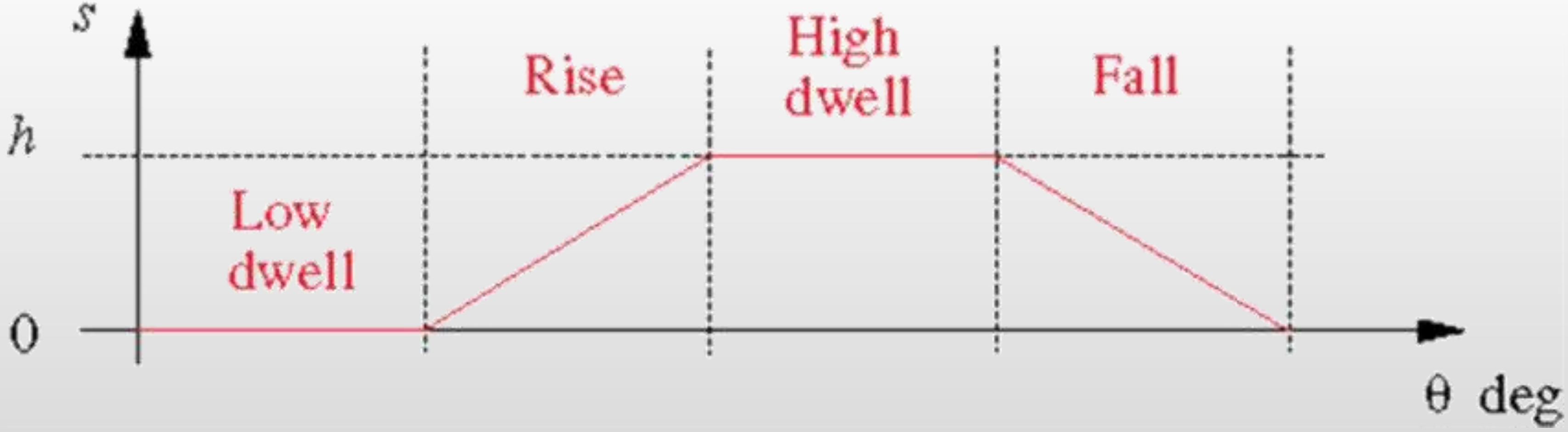
- **SCCA**: Sine–Constant–Cosine–Acceleration
  - It is a family of functions used to model the rise and fall
  - It can be defined by the same general equation
  - A change in numeric parameters yield different functions

$$s = f\left(b, c, d, \beta\right)$$



SCCA double dwell: Normalization

$$x = \frac{\theta}{\beta} \rightarrow \begin{cases} \theta \in [0, \beta] \\ x \in [0,1] \end{cases} \qquad y = \frac{s}{h} \rightarrow \begin{cases} s \in [0, h] \\ y \in [0,1] \end{cases}$$



SCCA double dwell: SVJA

- Divided into five zones 1 to 5 (five different forms)

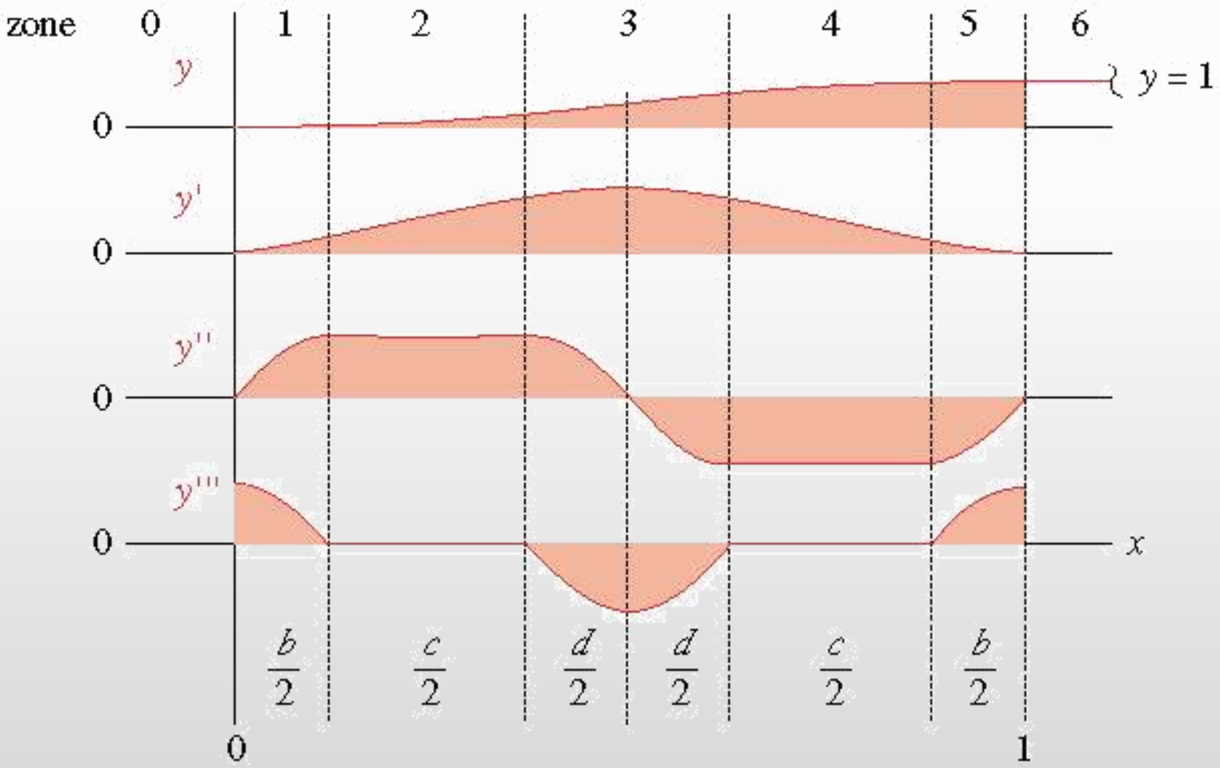
$$y_1 = f_1(b, c, d, x)$$

$$y_2 = f_2(b, c, d, x)$$

$$y_3 = f_3(b, c, d, x)$$

$$y_4 = f_4(b, c, d, x)$$

$$y_5 = f_5(b, c, d, x)$$



SCCA double dwell: SVJA – Zone 1

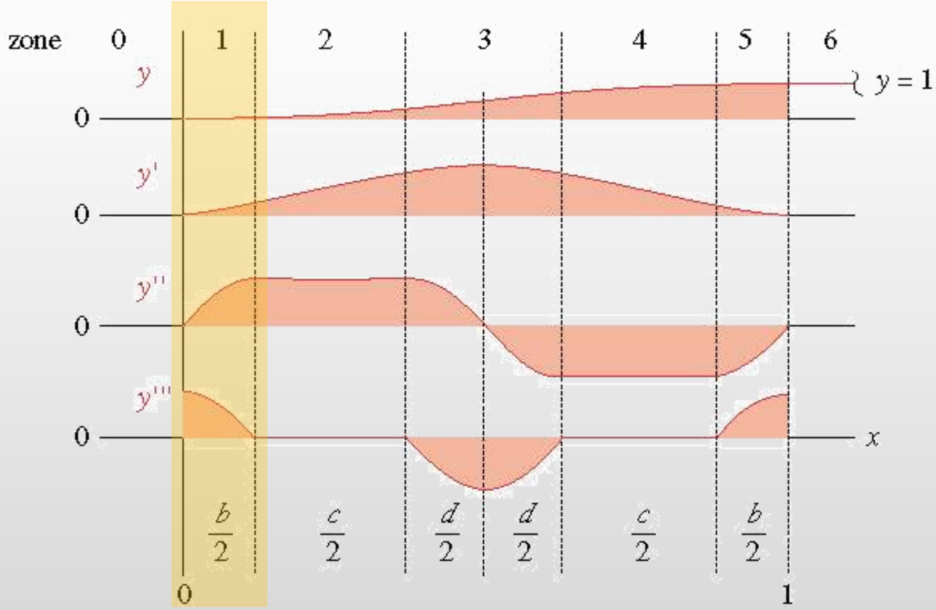
$$0 \le x \le \frac{b}{2} \qquad b \ne 0$$

$$y = C_a\left[\frac{b}{\pi}x - \left(\frac{b}{\pi}\right)^2 \sin\left(\frac{\pi}{b}x\right)\right]$$

$$y' = C_a\left[\frac{b}{\pi} - \frac{b}{\pi}\cos\left(\frac{\pi}{b}x\right)\right]$$

$$y'' = C_a \sin\left(\frac{\pi}{b}x\right)$$

$$y''' = C_a \frac{\pi}{b}\cos\left(\frac{\pi}{b}x\right)$$



SCCA double dwell: SVJA – Zone 2

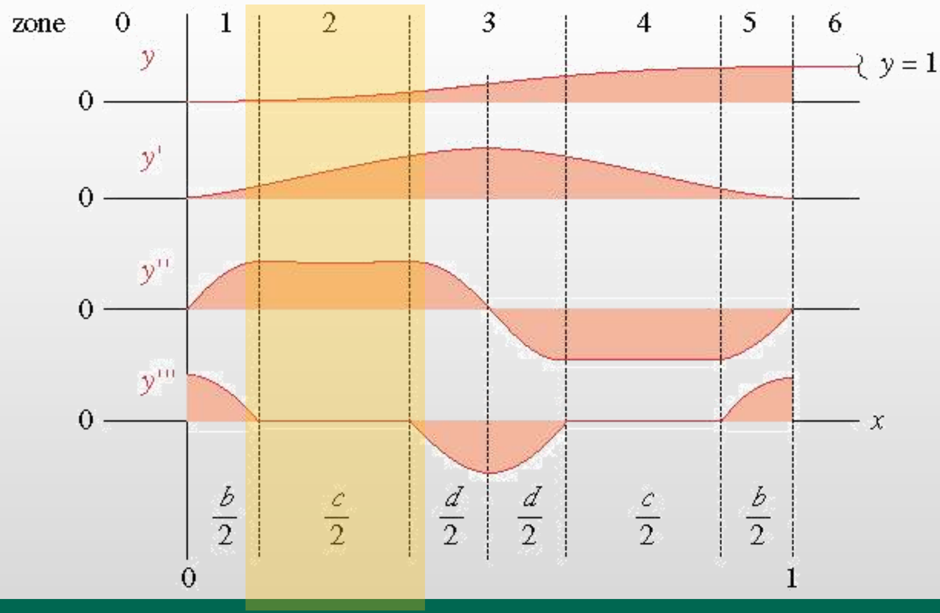
$$\frac{b}{2} \le x \le \frac{1-d}{2}$$

$$y = C_a\left[\frac{x^2}{2} + b\left(\frac{1}{\pi} - \frac{1}{2}\right)x + b^2\left(\frac{1}{8} - \frac{1}{\pi^2}\right)\right]$$

$$y' = C_a\left[x + b\left(\frac{1}{\pi} - \frac{1}{2}\right)\right]$$

$$y'' = C_a$$

$$y''' = 0$$



SCCA double dwell: SVJA – Zone 3

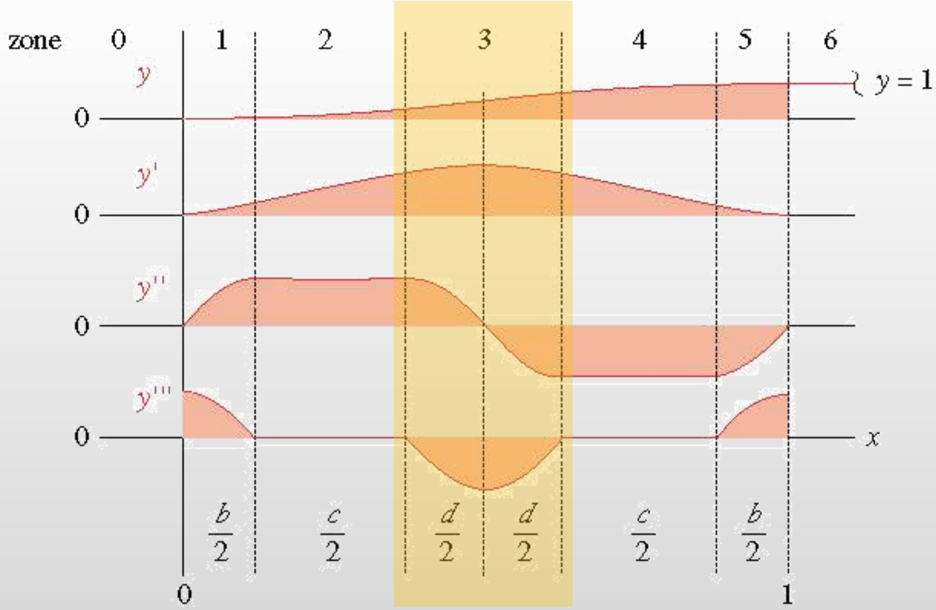
$$\frac{1-d}{2} \le x \le \frac{1+d}{2} \qquad d \ne 0$$

$$y = C_a\left\{\left(\frac{b}{\pi}+\frac{c}{2}\right)x+\left(\frac{d}{\pi}\right)^2+b^2\left(\frac{1}{8}-\frac{1}{\pi^2}\right)-\frac{(1-d)^2}{8}-\left(\frac{d}{\pi}\right)^2\cos\left[\frac{\pi}{d}\left(x-\frac{1-d}{8}\right)\right]\right\}$$

$$y' = C_a\left\{\frac{b}{\pi}+\frac{c}{2}+\frac{d}{\pi}\sin\left[\frac{\pi}{d}\left(x-\frac{1-d}{8}\right)\right]\right\}$$

$$y'' = C_a\cos\left[\frac{\pi}{d}\left(x-\frac{1-d}{8}\right)\right]$$

$$y''' = -C_a\frac{\pi}{d}\sin\left[\frac{\pi}{d}\left(x-\frac{1-d}{8}\right)\right]$$



SCCA double dwell: SVJA – Zone 4

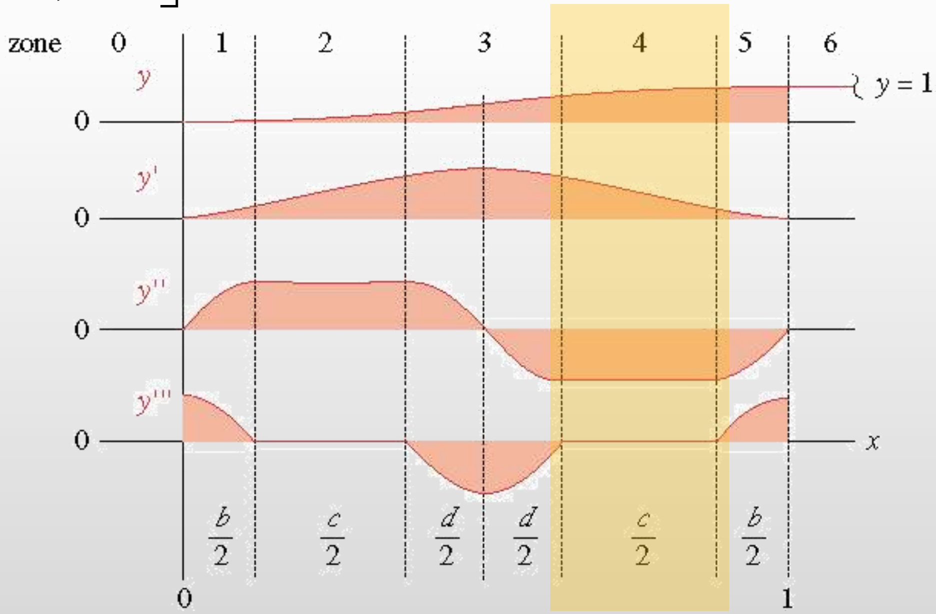
$$\frac{1+d}{2} \le x \le \frac{2-b}{2}$$

$$y = C_a\left[-\frac{x^2}{2} + \left(\frac{b}{\pi} + 1 - \frac{b}{2}\right)x + \left(2d^2 - b^2\right)\left(\frac{1}{\pi^2} - \frac{1}{8}\right) - \frac{1}{4}\right]$$

$$y' = C_a\left[-x + \frac{b}{\pi} + 1 - \frac{b}{2}\right]$$

$$y'' = -C_a$$

$$y''' = 0$$



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SCCA double dwell: SVJA – Zone 5

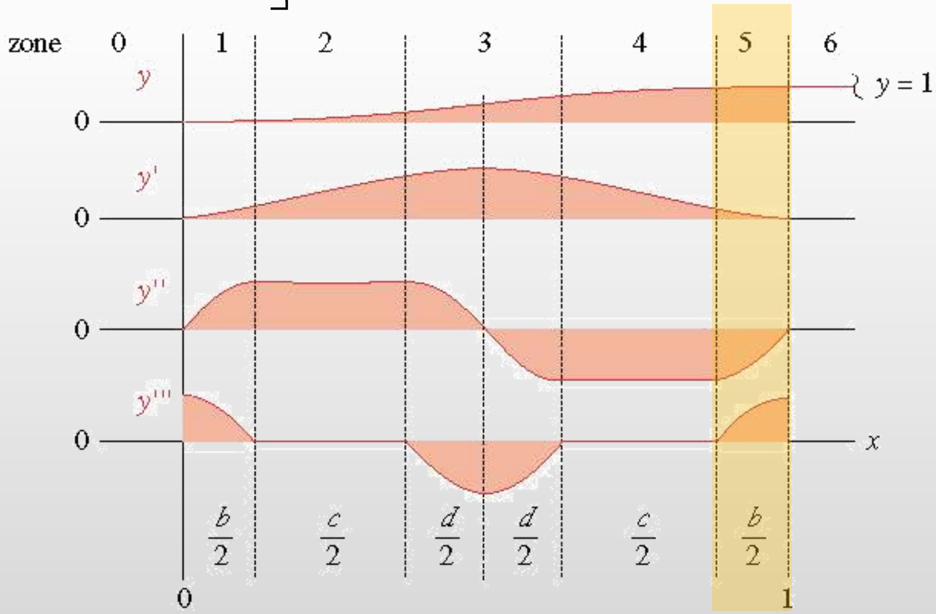
$$1-\frac{b}{2}\le x\le 1 \qquad b\neq 0$$

$$y=C_a\left[\frac{b}{\pi}x+\frac{2\left(d^2-b^2\right)}{\pi^2}+\frac{\left(1-b^2\right)-d^2}{4}-\left(\frac{b}{\pi}\right)^2\sin\left(\frac{\pi}{b}(x-1)\right)\right]$$

$$y'=C_a\left[\frac{b}{\pi}-\frac{b}{\pi}\cos\left(\frac{\pi}{b}(x-1)\right)\right]$$

$$y''=C_a\sin\left(\frac{\pi}{b}(x-1)\right)$$

$$y'''=C_a\frac{\pi}{b}\cos\left(\frac{\pi}{b}(x-1)\right)$$



SCCA double dwell – Coefficients

Velocity coefficient: $x = 0.5 \quad C_V = C_a\left(\frac{b+d}{\pi} + \frac{c}{2}\right)$

Maximum velocity

Acceleration coefficient: $x = 1,\ y = 1 \quad \rightarrow \quad C_a = \frac{4\pi^2}{(\pi^2 - 8)(b^2 - d^2) - 2\pi(\pi - 2)b + \pi^2}$

Maximum acceleration

Jerk coefficient: $x = 0 \quad C_j = C_a \frac{\pi}{b}$

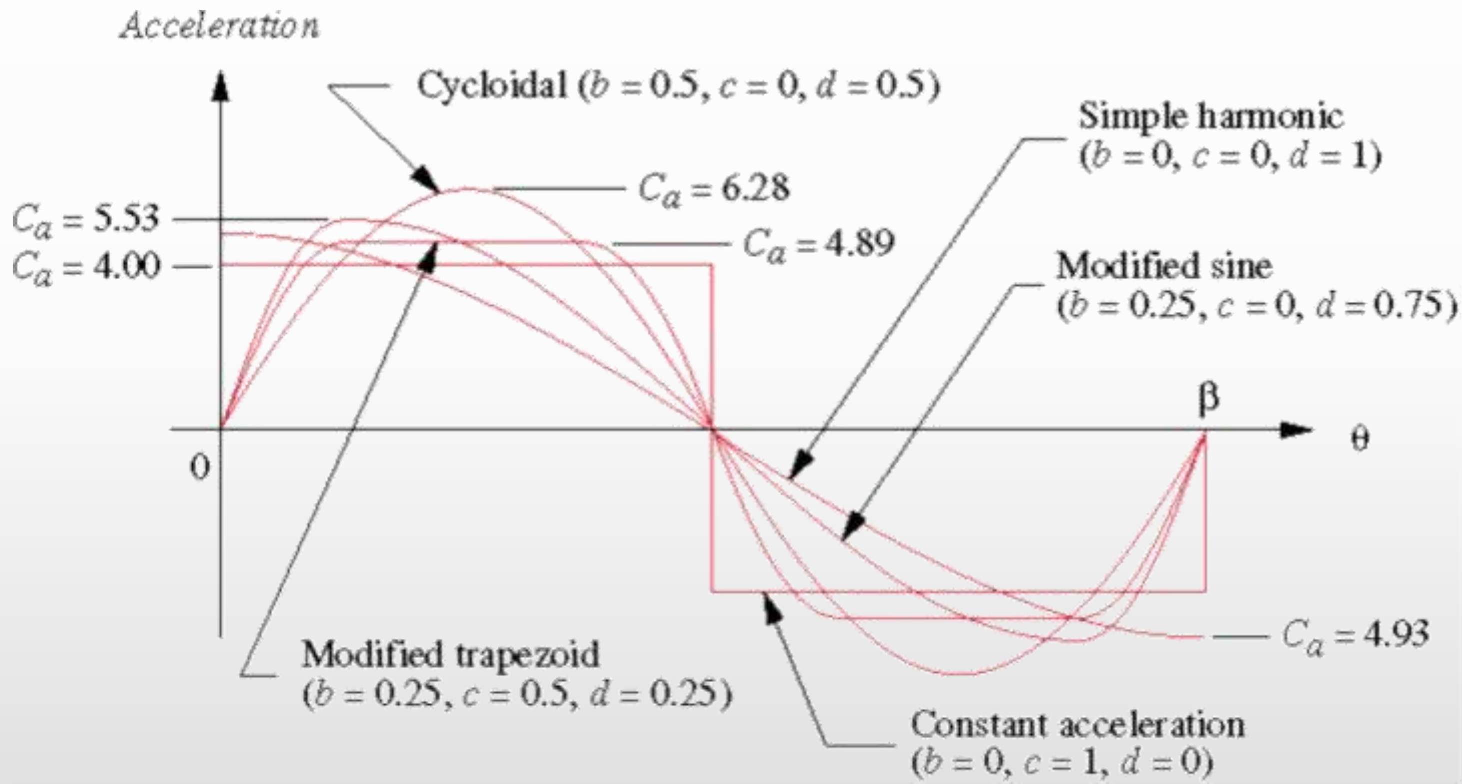
Maximum jerk

SCCA double dwell – Coefficients values

TABLE 8-2 Parameters and Coefficients for the SCCA Family of Functions

Function	b	c	d	C_v	C_a	C_j
constant acceleration	0.00	1.00	0.00	2.0000	4.0000	infinite
modified trapezoid	0.25	0.50	0.25	2.0000	4.8881	61.426
simple harmonic	0.00	0.00	1.00	1.5708	4.9348	infinite
modified sine	0.25	0.00	0.75	1.7596	5.5280	69.466
cycloidal displacement	0.50	0.00	0.50	2.0000	6.2832	39.478

Acceleration plot comparison



Avoid Constant acceleration or Simple harmonic

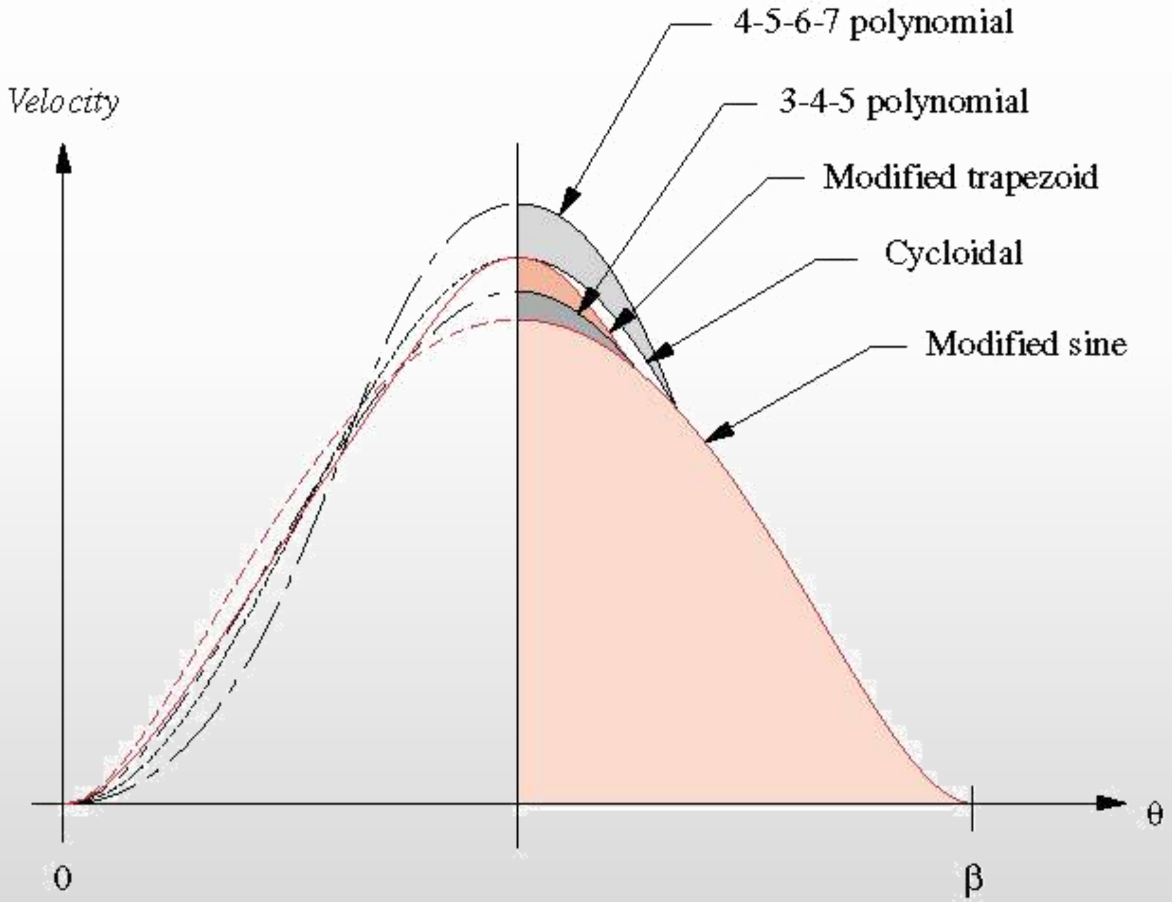
Cam Design Approach

- Design a Cam to minimize acceleration (use Mod. Trap.)
- Design a Cam to minimize velocity (use Mod. Sine.)

TABLE 8-2 Factors for Peak Velocity and Acceleration of Some Cam Functions

Function	Max. Veloc.	Max. Accel.	Max. Jerk	Comments
Constant accel.	2.000 h/β	4.000 h/β^2	infinite	∞ jerk - not acceptable.
Harmonic disp.	1.571 h/β	4.945 h/β^2	infinite	∞ jerk - not acceptable.
Trapezoid accel.	2.000 h/β	5.300 h/β^2	44 h/β^3	Not as good as mod. trap.
Mod. trap. accel.	2.000 h/β	4.888 h/β^2	61 h/β^3	Low accel but rough jerk.
Mod. sine. accel.	1.760 h/β	5.528 h/β^2	69 h/β^3	Low veloc - good accel.
3-4-5 Poly. disp.	1.875 h/β	5.777 h/β^2	60 h/β^3	Good compromise.
Cycloidal disp.	2.000 h/β	6.283 h/β^2	40 h/β^3	Smooth accel. & jerk.
4-5-6-7 Poly. disp.	2.188 h/β	7.526 h/β^2	52 h/β^3	Smooth jerk-high accel.

Cam Design: Velocity



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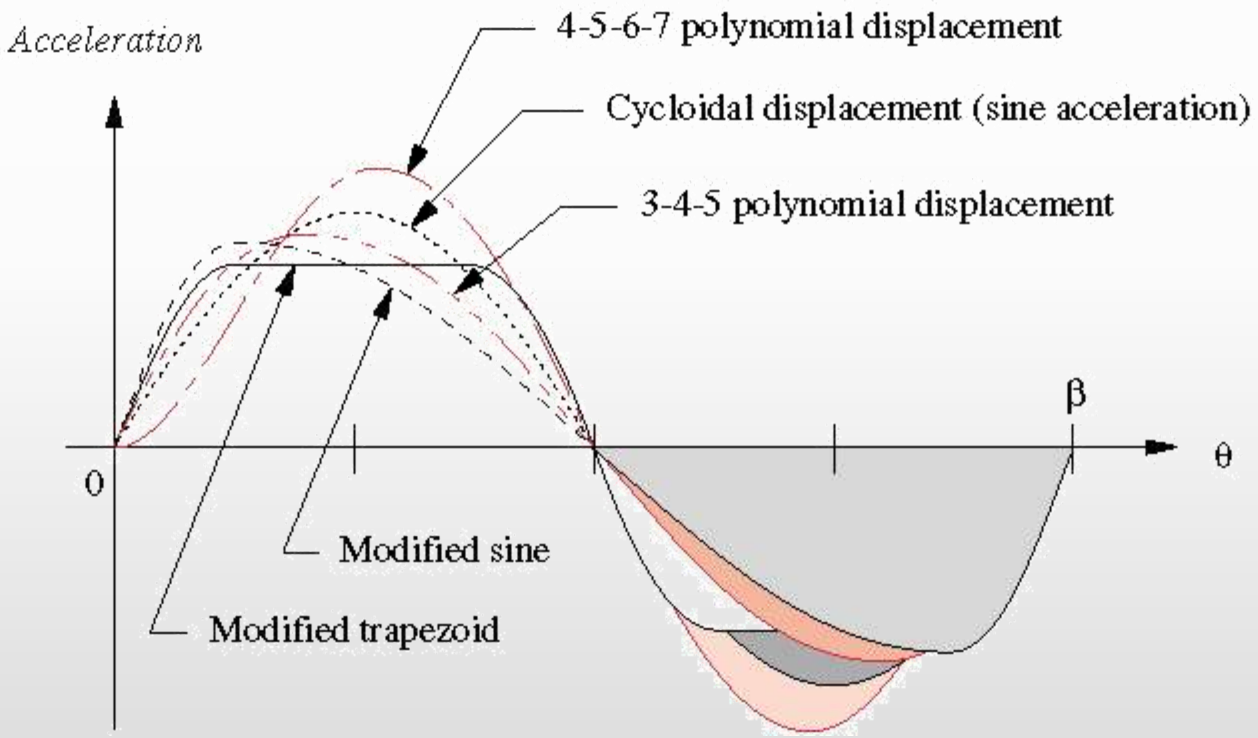
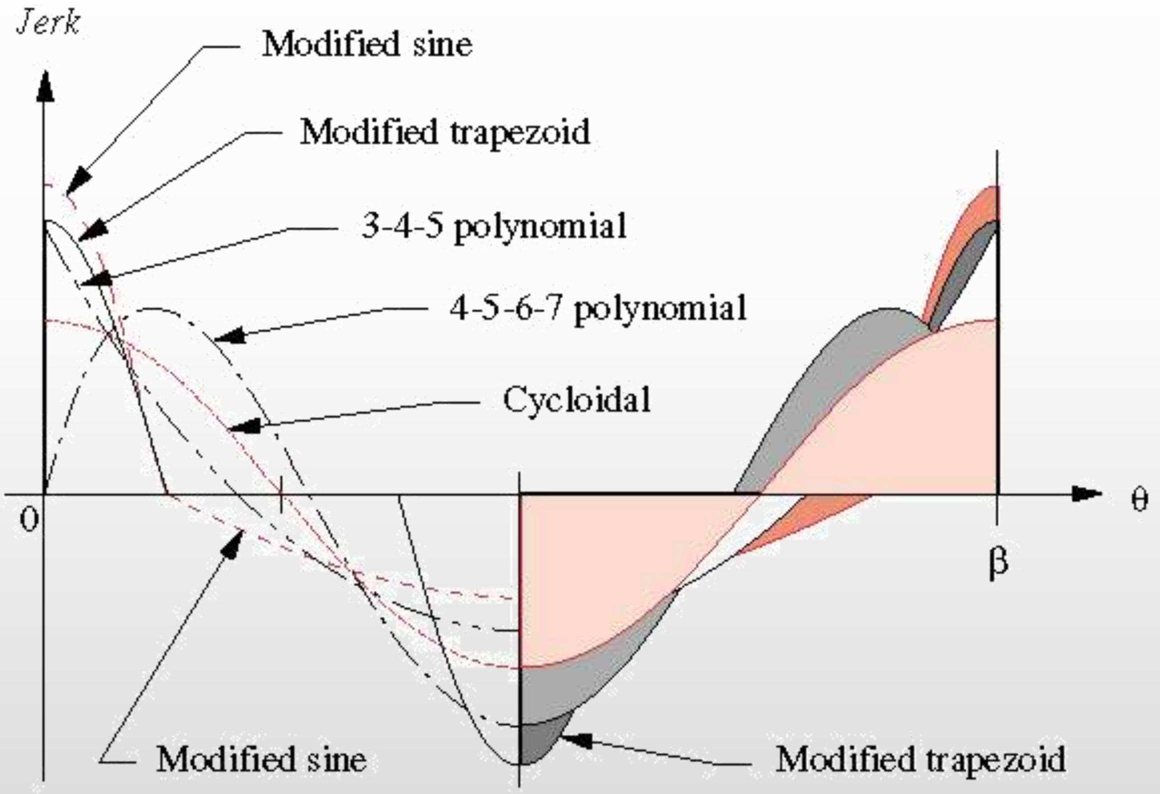
Cam Design: Acceleration



Cam Design: Jerk



Cam Design: Dwell

1. Use the same functions used for rise
2. Subtract the rise function s form the lift h
3. Negate the higher derivatives v, a and j

$$s_f = h - s_r$$

$$v_f = -v_r$$

$$a_f = -a_r$$

$$j_f = -j_r$$

Example

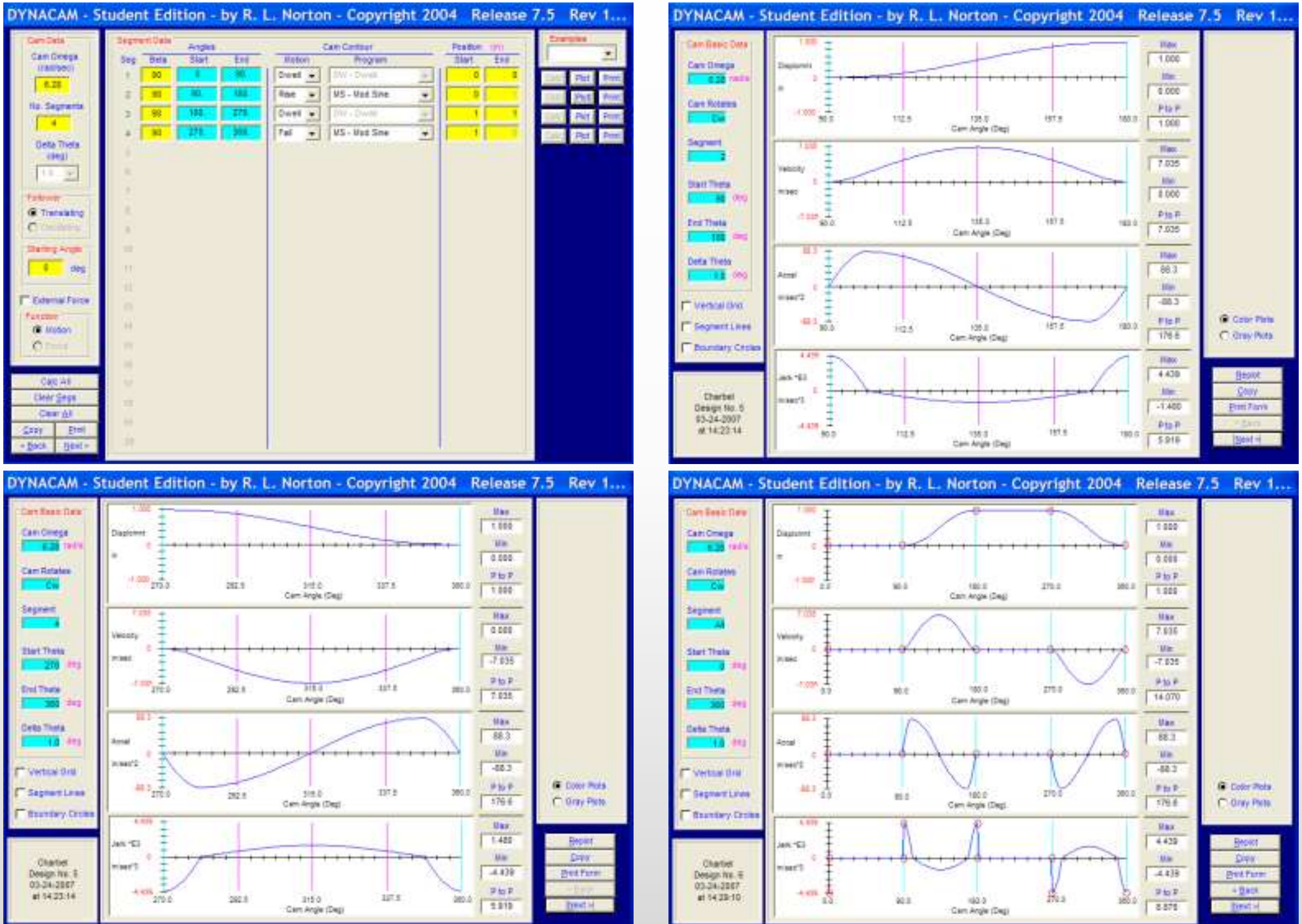
Consider the cam design specifications below:

Dwell	: at 0 displacement for 90°
Rise	: 1″ in 90°
Dwell	: at 1″ displacement for 90°
Fall	: 1″ in 90°
ω	: 2π rad/s

Minimize velocity → use modified Sine
Minimize acceleration → use modified Trapezoid

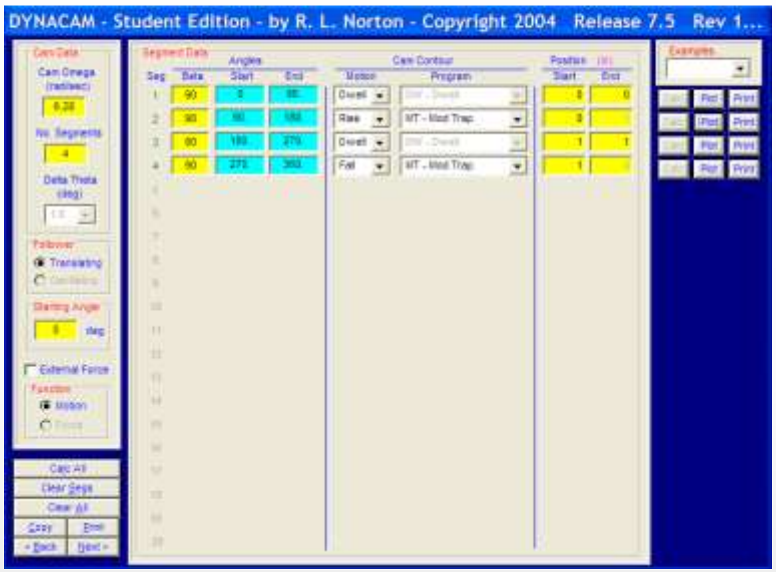
Example solution: DYNCAM

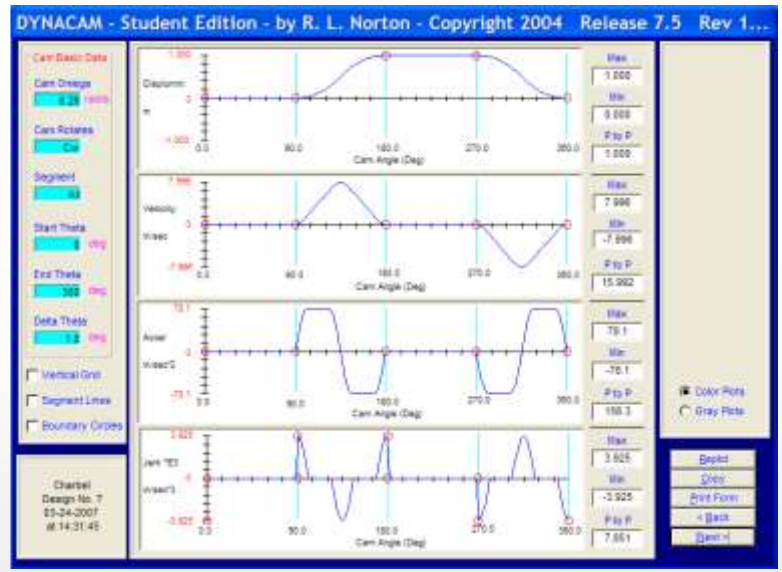
Minimize velocity → use modified Sine



Example solution: DYNCAM

Minimize velocity → use modified Trapezoid





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